Fuzzy Methods for Constructing Multi-Criteria Decision Functions

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Mixing Words and Mathematics

Building Decision Functions Using Information Expressed in Natural Language

Fuzzy Sets

A Fuzzy set F on a space X associates with each $x \in X$ a membership grade $F(x) \in [0, 1]$ indicating the degree to which the element x satisfies the concept being modeled by F

If F is modeling the concept ***tall*** and x is a person then F(x) is the degree to which x satisfies the concept tall

The Basics of MCDM With Fuzzy

- Representation of Criteria as Fuzzy Subset over the set of Decision Alternatives

- Here C(x) indicates the degree to which alternative C satisfies criteria C

- Allows Linguistic Formulation of Relationship Between Criteria Using Set Theoretic Operators to Construct Multi-Criteria Decision Function D

- The Resultant Multi-Criteria Decision Function D is itself a Fuzzy Subset over set of alternatives

- Selection of Preferred Alternative is Based on Alternatives Membership in D

Linguistic Expression of Multi-Criteria Decision Problem

Satisfy Criteria one and Criteria two and ....

- $D = C_1$ and C_2 and .... and C_n
- "and" as intersection of fuzzy sets
- $D = C_1 \cap C_2 \cap \ldots\ldots\ldots \cap C_n$
- $D(x) = \mathrm{Min}_j[C_j(x)]$
- Choose x^* with biggest $D(x)$

Importance Weighting in Multi-Criteria Decision Problem

- Associate with criteria C_j importance α_j

- $\alpha_j \in [0, 1]$ and $C_j(x) \in [0, 1]$

- $D(x) = \mathrm{Min}_j[(C_j(x))^{\alpha_j}]$

- $\mathrm{Min}[a, 1] = a$ & $(C_j(x))^0 = 1 \Rightarrow$ No effect of $\alpha_j = 0$

  $\mathrm{Min}[a, b]$: Smaller argument more effect

Anxiety In Decision Making

- Alternatives: $X = \{x_1, x_2, x_3, \ldots\ldots., x_q\}$

- Decision function D

  $D(x_j)$ is satisfaction by x_j

- x^* best alternative

- Anxiety associated with selection

$$Anx(D) = 1 - (D(x^*) - \frac{1}{q-1}\sum_{x_j \neq x^*} D(x_j))$$

Ordinal Scales

- $Z = \{z_0, z_1, z_3, \ldots\ldots\ldots, z_m\}$

  $z_i > z_k$ if $i > k$ (only ordering)

- Operations: Max and Min and Negation

  $Neg(z_j) = z_{m-j}$ (reversal of scale)

- Linguistic values generally only satisfy ordering

  Very High > High > Medium > Low > Very Low

- Often people only can provide information with this type of granulation

Ordinal Decision Making

Yager, R. R. (1981). A new methodology for ordinal multiple aspect decisions based on fuzzy sets. Decision Sciences 12, 589-600

- Criteria satisfactions and importances ordinal

- $\alpha_j \in Z$ and $C_j(x) \in Z$

- $D(x) = \mathrm{Min}_j[G_j(x)]$

  $G_j(x) = \mathrm{Max}(C_j(x), \mathrm{Neg}(\alpha_j))$

- $\alpha_j = z_0 \Rightarrow G_j(x) = z_m$ (No effect on D(x))

  $\alpha_j = z_m \Rightarrow G_j(x) = C_j(x)$

- Linguistic Expression: Satisfy Criteria one **and** Criteria two **and** ....

$$D = C_1 \text{ and } C_2 \text{ and } \ldots\ldots\ldots \text{ and } C_n$$

$$D = C_1 \cap C_2 \cap \ldots\ldots\ldots \cap C_n$$

$$D(x) = \mathrm{Min}_j[C_j(x)]$$

- Linguistic Expression: Satisfy Criteria one **or** Criteria two **or** ....

$$D = C_1 \text{ or } C_2 \text{ or } \ldots\ldots\ldots \text{ or } C_n$$

$$D = C_1 \cup C_2 \cup \ldots\ldots\ldots \cup C_n$$

$$D(x) = \mathrm{Max}_j[C_j(x)]$$

Building M-C Decision Functions

- **Linguistic Expression**

Satisfy Criteria one **and** Criteria two

or

Satisfy Criteria one **or** two **and** criteria 3

or

Satisfy criteria 4 **and** Criteria 3 **or** Criteria 2

- **Mathematical Formulation**

$$D = (C_1 \cap C_2) \cup ((C_1 \cup C_2) \cap C_3) \cup (C_4 \cap (C_3 \cup C_2))$$

Generalizing "and" Operators

t-norm operators generalize "and" (Min)

- $T: [0, 1] \times [0, 1] \to [0, 1]$
  1. $T(a, b) = T(b, a)$ Commutative
  2. $T(a, b) \geq T(c, d)$ if $a \geq c$ & $b \geq d$ Monotonic
  3. $T(a, T(b, c)) = T(T(a, b), c)$ Associative
  4. $T(a, 1) = a$ one as identity

- Many Examples of t-norms

  $T(a, b) = \mathrm{Min}[a, b]$ $T(a, b) = a\, b$ (product)

  $T(a, b) = \mathrm{Max}(a + b - 1, 0)$

$$T(a, b) = \mathrm{Max}\left(1 - ((1 - a)^{\lambda} + (1 - b)^{\lambda})^{\frac{1}{\lambda}}, 0\right)$$

Family parameterized by λ

Generalizing "or" Operators

t-conorm operators generalize "or" (Max)

- S: $[0, 1] \times [0, 1] \rightarrow [0, 1]$
  1. $S(a, b) = S(b, a)$ Commutative
  2. $S(a, b) \geq S(c, d)$ if $a \geq c$ & $b \geq d$ Monotonic
  3. $S(a, S(b, c)) = S(S(a, b), c)$ Associative
  4. $S(a, 0) = a$ zero as identity

- Many Examples of t-norms

$S(a, b) = Max[a, b]$ $S(a, b) = a + b - a b$

$S(a, b) = Min(a + b, 1)$

$$S(a, b) = Min((a^{\lambda} + b^{\lambda})^{\frac{1}{\lambda}}, 1)$$

Family parameterized by λ

Alternative Forms of Basic M-C functions

- $D = C_1$ **and** C_2 **and** .... **and** C_n
- $D(x) = T_j[C_j(x)]$
- $D(x) = \Pi_j C_j(x)$ (product)
- $D = C_1$ **or** C_2 **or** .... **or** C_n
- $D(x) = S_j[C_j(x)]$
- $D(x) = \text{Min}(\Sigma_j C_j(x), 1]$ (Bounded sum)

- Use of families of t-norms enables a parameterized representation of multi-criteria decision functions

- This opens the possibility of learning the associated parameters from data

- C_1 C_2 C_3 C_4 D

C_1	C_2	C_3	C_4	D
.3	.5	1	.7	.5

Generalized Importance Weighted "anding"

- $D = C_1$ **and** C_2 **and** …….. **and** C_n
- Associate with criteria C_j importance α_j
- $D(x) = \mathbf{T}_j[G_j(x)]$

  $G_j(x) = \mathbf{S}(C_j(x), 1 - \alpha_j)$
- $D(x) = \mathrm{Min}_j[(\mathrm{Max}(C_j(x), 1 - \alpha_j))$

  $D(x) = \prod(\mathrm{Max}(C_j(x), 1 - \alpha_j)$

Generalized Importance Weighted "oring"

- $D = C_1$ **or** C_2 **or** .... **or** C_n
- Associate with criteria C_j importance α_j
- $D(x) = \mathbf{S}_j[H_j(x)]$

  $H(x) = T(C_j(x),\ \alpha_j)$

- $D(x) = \mathrm{Max}_j[\mathrm{Min}(\alpha_j,\ C_j(x))]$

  $D(x) = \mathrm{Max}_j[\alpha_j\ C_j(x)]$

  $D(x) = \mathrm{Min}(\sum_j \alpha_j C_j(x),\ 1]$

Some Observations

- If any $C_j(x) = 0$ then

  $\mathbf{T}(C_1(x), C_1(x), \ldots\ldots, C_1(x)) = 0$

- Imperative of this class of decision functions is

  All *criteria must be satisfied*

- If any $C_j(x) = 1$ then

  $\mathbf{S}(C_1(x), C_1(x), \ldots\ldots, C_1(x)) = 1$

- Imperative of this class of decision functions is

  At least one *criteria must be satisfied*

$$D(x) = \frac{1}{n} \sum_{j=1}^{n} C_j(x)$$

Mean Operators

- $\mathbf{M}: R^n \rightarrow R$

  1. Commutative
  2. Monotonic

     $\mathbf{M}(a_1, a_2, \ldots\ldots, a_n) \geq \mathbf{M}(b_1, b_2, \ldots\ldots, b_n)$ if $a_j \geq b_j$
  3. Bounded

     $\mathrm{Min}_j[a_j] \leq \mathbf{M}(a_1, a_2, \ldots\ldots, a_n) \leq \mathrm{Max}_j[a_j]$

(Idempotent: $\mathbf{M}(a, a, \ldots\ldots, a) = a$

- Many Examples of Mean Operators

$\mathrm{Min}_j[a_j]$, $\mathrm{Max}_j[a_j]$, Median, Average

OWA Operators

Choquet Aggregation Operators

Ordered Weighted Averaging Operators

OWA Operators

Yager, R. R. (1988). On ordered weighted averaging aggregation operators in multi-criteria decision making. IEEE Transactions on Systems, Man and Cybernetics 18, 183-190

OWA Aggregation Operators

- Mapping $F: R^n \rightarrow R$ with $F(a_1, \ldots.., a_n) = \sum_{j=1}^{n} w_j b_j$
  - b_j is the j^{th} largest of the a_j
  - weights satisfy: **1.** $w_j \in [0, 1]$ and **2.** $\sum_{j=1}^{n} w_j = 1$

- Essential feature of the OWA operator is the reordering operation, **nonlinear operator**

- Weights not associated directly with an argument but with the ordered position of the arguments

- $W = [w_1 \ w_2 \qquad w_n]$ called the **weighting vector**

- $B = [b_1 \ b_2 \qquad b_n]$ is **ordered argument vector**

- $F(a_1, \ldots.., a_n) = W\ B^T$

- If id(j) is index of jth largest of a_i then

$$F(a_1, \ldots.., a_n) = \sum_{j=1}^{n} w_j \, a_{id(j)}$$

$$a_{id(j)} = b_j$$

Form of Aggregation is Dependent Upon the Weighting Vector Used

OWA Aggregation is Parameterized by W

Some Examples

- W^*: $w_1 = 1$ & $w_j = 0$ for $j \neq 1$ gives

$$F^*(a_1, \ldots.., a_n) = \text{Max}_i[a_i]$$

- W_*: $w_n = 1$ & $w_j = 0$ for $j \neq n$ gives

$$F^*(a_1, \ldots.., a_n) = \text{Min}_i[a_i]$$

- W_N: $w_j = \frac{1}{n}$ for all j gives the simple average

$$F^*(a_1, \ldots.., a_n) = \frac{1}{n} \sum_{i=1}^{n} a_i$$

Attitudinal Character of an OWA Operator

- $\text{A-C}(W) = \dfrac{1}{n-1} \sum_{j=1}^{n} w_j \, (n-j)$

- Characterization of type of aggregation

- $\text{A-C}(W) \in [0, 1]$

- $\text{A-C}(W^*) = 1 \qquad \text{A-C}(W_N) = 0.5 \qquad \text{A-C}(W_*) = 0$

- Weights symmetric $(w_j = w_{n-j+1}) \Rightarrow \text{A-C}(W) = 0.5$

An A-C value near **one** indicates a bias toward the **larger** values in the argument (**Or-like /Max-like**)

An A-C value near **zero** indicates a bias toward the **smaller** values in the argument (**And-like /Min-like**)

An A-C value **near 0.5** is an indication of a **neutral** type aggregation

Measure of Dispersion an OWA Operator

- $\text{Disp}(W) = - \sum_{j=1}^{n} w_j \ln(w_j)$

- Characterization amount of information used

- $\text{Disp}(W^*) = \text{Disp}(W_*) = 0$ (Smallest value)

  $\text{A-C}(W_N) = \ln(n)$ (Largest value)

- Alternative Measure

  $$\text{Disp}(W) = \sum_{j=1}^{n} (w_j)^2$$

Some Further Notable Examples

- **Median**: if n is **odd** then $w_{\frac{n+1}{2}} = 1$

  if n is **even** then $w_{\frac{n}{2}} = w_{\frac{n}{2}+1} = \frac{1}{2}$

- **kth best:** $w_k = 1$ then $F^*(a_1, \ldots.., a_n) = a_{id(k)}$

- **Olympic Average**: $w_1 = w_n = 0$, other $w_j = \frac{1}{n-2}$

- **Hurwicz average**: $w_1 = \alpha$, $w_n = 1\text{-}\alpha$, other $w_j = 0$

OWA Operators Provide a Whole family of functions for the construction of mean like multi–Criteria decision functions

$$D(x) = F_W(C_1(x), C_2(x), ..., C_n(x))$$

Selection of Weighting Vector
Some Methods

1. Direct choice of the weights

2. Select a notable type of aggregation

3. Learn the weights from data

4. Use characterizing features

5. Linguistic Specification

Learning the Weights from Data

- Filev, D. P., & Yager, R. R. (1994). Learning OWA operator weights from data. Proceedings of the Third IEEE International Conference on Fuzzy Systems, Orlando, 468-473.

- Filev, D. P., & Yager, R. R. (1998). On the issue of obtaining OWA operator weights. Fuzzy Sets and Systems 94, 157-169.

- Torra, V. (1999). On learning of weights in some aggregation operators: the weighted mean and the OWA operators. Mathware and Softcomputing 6, 249-265

Algorithm for Learning OWA Weights

- Express OWA weights as $w_j = \dfrac{e^{\lambda_j}}{\sum_{k=1}^{n} e^{\lambda_k}}$
- Use data of observations to learn λ_i

  $(a_1, \quad , a_n)$ and aggregated value d
- Order arguments to get b_j for $j = 1$ to n
- Using current estimate of weights calculate

$$\hat{d} = \sum_{j=1}^{n} w_j \, b_j$$

- Updated estimates of λ_j

$$\lambda'_j = \lambda_j - \alpha w_j (b_i - \hat{d}) \; (\hat{d} - d)$$

Using Characterizing Features

- $$A\text{-}C(W) = \frac{1}{n-1} \sum_{j=1}^{n} w_j (n - j)$$

- $A\text{-}C(W) = 1$ "orlike"

  $A\text{-}C(W) = 0$ "andlike"

- $\alpha \in [0, 1]$ degree of "orness"

- Determine W with specified α

O'Hagan Method

- Specify α and determine weights to maximize the dispersion

- $$\text{Max } -\sum_{j=1}^{n} w_j \ln(w_j)$$

  such that

  1. $$\frac{1}{n-1}\sum_{j=1}^{n} w_j\,(n-j) = \alpha$$
  2. $$\sum_{j=1}^{n} w_j = 1$$
  3. $$w_j \geq 0$$

Linguistic Specification of Weights

1. Linguistically specify aggregation imperative of multiple criteria

2. Translate linguistic imperative into Fuzzy Set

3. Use fuzzy set to determine OWA weights

Computing with Information Specified in a Natural Language

Quantifier Guided Criteria Aggregation

- D = Min: **All** criteria must be satisfied

  D = Max: ***At least one*** *criteria must be satisfied*

  "Quantifier" criteria must be satisfied

- Other examples of linguistic quantifiers:

  most, almost all, at least half

  only a few, at least 1/3

- Monotonic quantifiers

Representation of Linguistic Quantifier

- Represent quantifier as fuzzy subset Q on unit interval

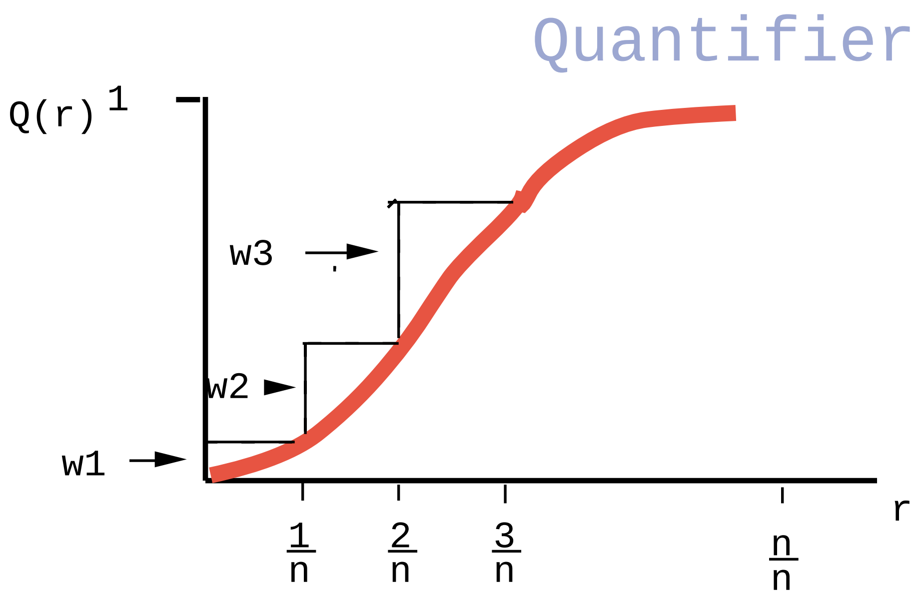
- Q(r) is the degree the proportion r satisfies the concept of the quantifier

- $Q : [0, 1] \rightarrow [0, 1]$

  1. $Q(0) = 0$
  2. $Q(1) = 1$
  3. $Q(r) \geq Q(p)$ if $r > p$

BUM Function

Obtaining OWA Weights from Quantifier



- $w_j = Q(\frac{j}{n}) - Q(\frac{j-1}{n})$

Functionally Guided Criteria Aggregation

- Specify a Bum function f: [0, 1] $\rightarrow$ [0, 1]

  1. $f(0) = 0$
  2. $f(1) = 1$
  3. $f(r) \geq f(p)$ if $r > p$

- $w_j = f(\frac{j}{n}) - f(\frac{j-1}{n})$

- Linear function $f(r) = r$ Quantifier $\Leftrightarrow$ Some

  $w_j = \frac{1}{n}$

Importance Weighted OWA Multi-Criteria Decision Functions

- Importance v_i associated criteria C_i

- Aggregation Agenda

  Quantifier Important Criteria are Satisfied

  Most Important Criteria are Satisfied

- $D(x) = F_{Q/V}(a_1, a_2, \ldots, a_n)$

  $$a_i = C_i(x)$$

Calculation of $D(x) = F_{Q/V}(a_1, a_2, \ldots\ldots, a_n)$

- Order the criteria satisfactions the a_i
- $a_{id(j)}$ is j^{th} largest & $v_{id(j)}$ its importance
- Calculate $S_j = \sum_{k=1}^{j} v_{id(k)}$ & $T = S_n = \sum_{k=1}^{n} v_{id(k)}$
- Determine OWA Weights

$$\tilde{w}_j = Q\left(\frac{S_j}{T}\right) - Q\left(\frac{S_{j-1}}{T}\right)$$

- $D(x) = \sum_{j=1}^{n} \tilde{w}_j \, a_{id(j)}$

Some Methods of Obtaining Importances

- Fixed Specified Value

- Determined by Property of Alternative

$$v_j = E(x)$$

- Dependent upon Other Attribute in Aggregation

$$v_j = C_k(x)$$

Induces a prioritization

- Rule Based

Concept Based Hierarchical Formulation of Multi-Criteria Decision Functions Using OWA Operators

Definition of a Concept

- Concept is more abstract criteria

$$Con \equiv \langle C_1, C_2, \ldots., C_n : V : Q \rangle.$$

- C_i are a collection of measurable criteria
- Q is an OWA Aggregation Imperative
- V vector where v_i is importance of C_i in concept
- $Con(x) = F_{Q/V}(C_1(x), C_2(x), \ldots., C_n(x))$

Concepts with Concepts as Components

$$\mathbf{Con} = <Con_1, Con_2, ..., Con_q : V : Q>.$$

$$Con(x) = F_{Q/V}(Con_1(x), Con_2(x),...., Con_q(x))$$

Multi-Criteria Decision Function Viewed as Concept

Allows hierarchical structure for the multi-criteria decision functions

Decision function:

$(C_1 \text{ and } C_2 \text{ and } C_3) \text{ or } (C_3 \text{ and } C_4)$

Represent as concept: $<Con_1, Con_2 : V: Q>$.

Here Q is ***or*** and $V = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Additionally

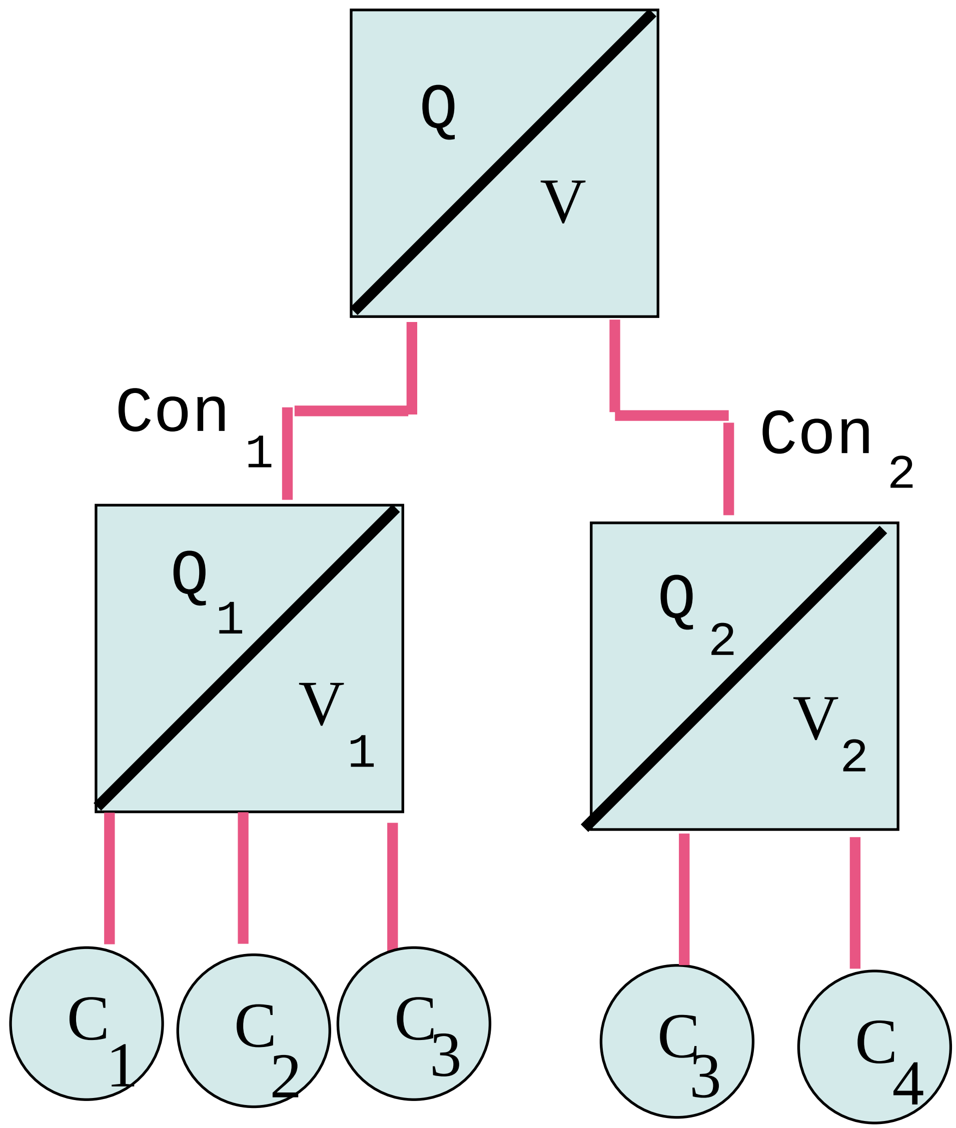
$Con_1 = <C_1, C_2, C_3: V_1: Q_1>$

$Con_2 = <C_3, C_4 : V_2: Q_2>$

Where $Q_1 = Q_2 =$ ***all***

$V_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Hierarchical Formulation



Ordinal OWA Operator

- $Z = \{z_0, z_1, z_3, \ldots\ldots\ldots, z_m\}$ ordinal scale
- Mapping $F\!: Z^n \to Z$ with

$$F(a_1, \ldots\ldots, a_n) = \mathrm{Max}_j[w_j \wedge b_j]$$

  - b_j is the jth largest of the a_j
  - weights satisfy: 1. $w_j \in Z$

    2. $w_i \geq w_k$ if $i > j$

    3. $w_n = z_m$

- Allows mean like M-C decision functions with ordinal information

Multi-Criteria Decision Functions Using Choquet Aggregation Operators

- Provides wide class of M-C decision functions

- $C = \{C_1, C_2, \ldots\ldots.., C_n\}$ "set of all criteria"

- Requires specification of monotonic measure μ over set of criteria

- $D(x) = G_\mu(a_1, a_2, \ldots.., a_n)$

$$a_i = C_i(x)$$

Set Measure μ

- For any subset A of criteria, $\mu(A)$ indicates the acceptability of a solution that satisfies all the criteria in A

- $\mu: 2^{C} \rightarrow [0, 1]$ (subsets of C into the unit interval)
    1. $\mu(\varnothing) = 0$
    2. $\mu(C) = 1$
    3. $\mu(A) \geq \mu(B)$ if $B \subset A$

- $\mu(\varnothing) = 0$ & $\mu(A) = 1$ "any criteria is okay"
  $\mu(C) = 1$ & $\mu(A) = 0$ "all criteria are needed"

Evaluation of Choquet M-C Decision Function

- $D(x) = G_\mu(a_1, a_2, \ldots.., a_n)$ $\quad a_i = C_i(x)$

- Order criteria satisfactions $\Rightarrow$ $a_{id(j)}$ is j^{th} largest

- $H_j = \{C_{id(k)} \mid k = 1 \text{ to } j\}$, j most satisfied criteria

- $w_j = \mu(H_j) - \mu(H_{j-1})$

- $D(x) = G_\mu(a_1, a_2, \ldots.., a_n) = \sum_{j=1}^{n} w_j \, a_{id(j)}$

Uninorms

- t-norm operators

$$T(a_1, a_2, \ldots.., a_n) = T(a_1, a_2, \ldots.., a_n, 1)$$

Identity is One

$$T(a_1, a_2, \ldots.., a_n) \geq T(a_1, a_2, \ldots.., a_n, a_{n+1})$$

- t-conorm operators

$$S(a_1, a_2, \ldots.., a_n) \leq S(a_1, a_2, \ldots.., a_n, a_{n+1})$$

Identity is Zero

$$T(a_1, a_2, \ldots.., a_n) = T(a_1, a_2, \ldots.., a_n, 0)$$

- Uninorm operators

Identity is **e** $\in$ [0, 1]

Uninorm operators with identity **e**

For $a_{n+1} < \mathbf{e}$

$$U(a_1, a_2, \ldots\ldots, a_n) \le U(a_1, a_2, \ldots\ldots, a_n, a_{n+1})$$

For $a_{n+1} = \mathbf{e}$

$$U(a_1, a_2, \ldots\ldots, a_n) = U(a_1, a_2, \ldots\ldots, a_n, \mathbf{e})$$

For $a_{n+1} > \mathbf{e}$

$$U(a_1, a_2, \ldots\ldots, a_n) \ge U(a_1, a_2, \ldots\ldots, a_n, a_{n+1})$$

M-C Decision Functions Using Uninorms

- Multi-Criteria Decision Function

  $D(X) = U(C_1(x), \ldots\ldots, C_n(x))$

- Criteria with satisfaction greater then **e** have positive effect while those less then **e** have negative effect

- Introduces bipolar scale

- **e** acts like "0" in a zero in simple addition

Multi-Criteria Decision Functions Using Fuzzy Systems Modeling

- Set of Criteria C_1, C_2, ..., C_n
- Describe Decision Function $D(x)$
- If $S.C_1$ is A_{11} and ... $S.C_n$ is A_{1n} then $D(x)$ is d_1

  If $S.C_1$ is A_{m1} and ... $S.C_n$ is A_{mn} then $D(x)$ is d_m

- A_{ij} is fuzzy subset of unit interval

  d_i value in the unit interval

  $S.C_j$ denotes variable "satisfaction of Criteria C_j"

Evaluation of Decision Function by Alternative

- Determine Satisfaction of Rule i by alternative x

$$r_i(x) = \prod_{j=1}^{n} A_{ij}(C_j(x))$$

- Obtain overall satisfaction

$$D(x) = \frac{\sum_{i=1}^{m} r_i(x)\ d_i}{\sum_{i=1}^{m} r_i(x)}$$

Multi-Criteria Decision Choice Procedure

Select x* such that

$$D(x^*) = \mathrm{Max}[D(x_j)]$$

Random Experiment Decisions
RED CHOICE

Calculate

$$b_j = \frac{D(x_j)}{\mathrm{Max}_i[D(x_i)]} \text{ and } p_j = \frac{(b_j)^\lambda}{\sum_{i=1}^{n} (b_i)^\lambda}$$

Perform random experiment with P_j as probability of x_j as outcome

Select outcome of experiment as choice

If $\lambda \to \infty$ then select x* (alternative with Max satisfaction

If $\lambda = 0$ then all P_j are equal

If $\lambda = 1$ then $P_j = \dfrac{D(x_j)}{\sum_i D(x_i)}$

λ is a reflection of confidence in Multi-Criteria Decision function D

Formulation of D and Criteria Valuations

Evaluating Criteria Satisfaction $C_j(x)$

- Scalar Number: $C_j(x) = 0.7$
- Ordinal Value: $C_j(x)$ = medium
- Interval Valued : $C_j(x) = [0.3, 0.7]$
- Fuzzy Set Valued: $C_j(x)$ is a fuzzy subset of [0, 1]
- Intuitionistic Values: $C_j(x) = (a, b)$ $/a + b \leq 1$

  a degree satisfaction/b degree not satisfaction
- Probabilistic Values: $C_j(x)$ is Probability distribution on [0, 1]

THE END

Lexicographically Prioritized Multicriteria Decisions Using Scoring Functions

Multi-Criteria Decision Problem

- Collection of criteria $C = \{C_1, ..., C_n\}$
- Set of alternatives $X = \{x_1, ..., x_m\}$.
- $C_i(x)$ as a value in the unit interval
- Overall satisfaction of alternative to criteria

$$C(x) = \sum_i w_i \, C_i(x)$$

- Weighted Aggregation of criteria satisfactions

Properties of Importance Weights

- $w_i \in [0, 1]$
- C(x) is called a **weighted scoring function**
- C(x) is monotonic in $C_i(x)$
- **Special case**: w_i sum to 1

  C(x) is called a **weighted averaging function**

  $\mathrm{Min}_i[C_i(x)] \le C(x) \le \mathrm{Max}_i[C_i(x)]$ (Bounded)

These weighted aggregation operators allow tradeoffs between criteria.

We can compensate for decrease of Δ in satisfaction to criteria C_i by gain $w_k/w_i\ \Delta$ in satisfaction to criteria C_k.

In some applications we may have a **lexicographic** ordering of the criteria and do not want to allow this kind of compensation between criteria.

Child Bicycle Selection Problem

- Selecting bicycle for child using criteria of **safety** and **cost**
- However any bicycle we select must be safe
- **We do not want poor safety to be compensated for by very low cost**.
- Before considering cost must be sure the bicycle is safe.
- A **lexicographic** induced prioritization ordering of criteria.
- Safety has a higher priority.

- In organizational decision making criteria desired by superiors generally, have a higher priority then those of their subordinates. The subordinate must select from among the solutions acceptable to the superior.

- Air traffic controller decisions involve a prioritization of considerations with passenger safety usually at the top.

WHAT IS NEEDED

An aggregation operator that can handle lexicographically induced priority between the criteria

Solution Imperative

- Use importance weights

- Importance weight of lower priority criteria based on satisfaction to higher priority criteria

- Effectively prevents satisfaction of lower priority criteria from compensating for poor satisfaction to higher priority criteria.

Prioritized Scoring

Operator

Problem Formulation

- Collection of criteria partitioned into q distinct categories

$$H_1, H_2, \ldots, H_q$$

- $H_i = \{C_{i1}, C_{i2}, \ldots, C_{in_i}\}$: C_{ij} are the criteria in category H_i
- A prioritization between these categories

$$H_1 > H_2, \ldots > H_q$$

- Criteria in H_i have a higher priority than those in H_k if $i < k$
- Criteria in the same category have the same priority
- Total number of criteria is n

Prioritized Scoring Operator
PS Operator

- Alternative $x \in X$
- $C_{ij}(x) \in [0, 1]$ is x satisfaction to criteria C_{ij}.
- $C(x)$ overall score for alternative x
- **P**rioritized **S**coring (PS) operator

$$C(x) = \sum_{i=1}^{q} \left(\sum_{j=1}^{n_i} w_{ij} C_{ij}(x) \right)$$

- Weights used to enforce the priority relationship
- Weights will be dependent on x

Determination of Weights

- For each category H_i we calculate $S_i = \text{Min}_j[C_{ij}(x)]$
- S_i is the value of the least satisfied criteria in category H_i
- $S_0 = 1$ by convention
- Calculate

$$T_i = \prod_{k=1}^{i-1} S_k \qquad (T_3 = S_0 S_1 S_2)$$

- Set $u_{ij} = T_i$
- Use $w_{ij} = u_{ij}$

Properties of the weights

- Criteria in same category have same weight

$$w_{ij} = T_i$$

- Criteria in top category have weight 0ne

$T_1 = 1$ (Criteria in H_1 have weight 1)

- Lower priority criteria smaller weights

$$T_i \geq T_k \text{ for } i < k$$

- If $S_i = 0$ then $w_{kj} = 0$ for $k > i$ (Contribution blocked)

Effective Prioritized Scoring Operator

$$C(x) = \sum_{i=1}^{q} T_i\left(\sum_{j=1}^{n_i} C_{ij}(x)\right)$$

T_i decreases as i increases

Low satisfaction for higher priority criteria

blocks contribution by low priority criteria

Manifests Fundamental Feature of the Prioritization

Poor satisfaction to any higher criteria reduces the ability for compensation by lower priority criteria.

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Example

- $H_1 = \{C_{11}, C_{12}\}$, $H_2 = \{C_{21}\}$, $H_3 = \{C_{31}, C_{32}, C_{33}\}$

  $H_4 = \{C_{41}, C_{42}\}$
- $C_{11}(x) = 0.7$, $C_{12}(x) = 1$, $C_{21}(x) = 0.9$, $C_{31}(x) = 0.8$

  $C_{32}(x) = 1$, $C_{33}(x) = 0.2$, $C_{41}(x) = 1$, $C_{42}(x) = 0.9$
- $S_1 = \mathrm{Min}[C_{11}(x), C_{12}(x)] = 0.7$

  $S_2 = \mathrm{Min}[C_{21}(x)] = 0.9$

  $S_3 = \mathrm{Min}[C_{31}(x), C_{32}(x), C_{33}(x)] = 0.2$

  $S_4 = \mathrm{Min}[C_{41}(x), C_{42}(x)] = 0.9$
- $T_1 = 1$, $T_2 = S_1T_1 = 0.7$, $T_3 = S_2T_2 = 0.63$

  $T_4 = S_3T_3 = 0.12$
- $w_{11} = w_{12} = 1$, $w_{21} = 0.7$, $w_{31} = w_{32} = w_{33} = 0.63$

  $w_{41} = w_{42} = 0.12$
- $C(x) = \sum w_{ij}C_{ij}(x) = 3.82$

Basic Features of the PS Operator

- Importance weights of a criterion depend on the satisfaction to higher priority criteria
- Lower priority criteria only contribute to the score of alternatives satisfying higher priority criteria
- Lower priority criteria used to distinguish between alternatives satisfying higher priority criteria
- Importance weights will be different across alternatives.

Why have we chosen this scoring type operator rather then an averaging operator which simply requires that we normalize the weights ?

In this case of partial ordering of the criteria (more the one criteria in each category) performing this normalization does not always guarantee a monotonic aggregation

Example 1

- $H_1 = \{C_{11}, C_{12}, C_{13}, C_{14}\}$ $H_2 = \{C_{21}, C_{22}, C_{23}\}$
- $C_{11}(x) = C_{12}(x) = C_{13}(x) = 1, C_{14}(x) = 0$
  $C_{21}(x) = C_{22}(x) = C_{23}(x) = 0.$
- $S_1 = 0$ hence $T_1 = 1$ and $T_2 = 0$.
- $u_{1j} = T_1 =$ and $u_{2j} = T_2 = 0$ and hence $\sum_{ij} u_{ij} = 4$
- Applying Normalization

$w_{1j} = 1/4$ for $j = 1$ to 4 $w_{2j} = 0$ for $j = 1$ to 3

- $C(x) = 1/4(C_{11}(x) + C_{12}(x) + C_{13}(x) + C_{14}(x)) = 0.75$

Example 2

- $H_1 = \{C_{11}, C_{12}, C_{13}, C_{14}\}$ $\quad\quad$ $H_2 = \{C_{21}, C_{22}, C_{23}\}$
- $C_{11}(x) = C_{12}(x) = C_{13}(x) = 1$, $\mathbf{C_{14}(x) = 1}$
  $C_{21}(x) = C_{22}(x) = C_{23}(x) = 0$.
- $\mathbf{S_1 = 1}$ hence $T_1 = 1$ and $\mathbf{T_2 = 1}$.
- $u_{1j} = T_1 = 1$ and $\mathbf{u_{2j} = T_2 = 1}$ and hence $\sum_{ij} u_{ij} = \mathbf{7}$
- Applying Normalization
  $w_{1j} = \mathbf{1/7}$ for $j = 1$ to 4 $\quad\quad$ $w_{2j} = \mathbf{1/7}$ for $j = 1$ to 3
- $C(x) = \mathbf{1/7}(C_{11}(x) + C_{12}(x) + C_{13}(x) + C_{14}(x)) = \mathbf{0.57}$
- $\mathbf{0.57 < 0.75}$

Prioritized Scoring Operator Respects the Monotonicity

For example 1

- $w_{1j} = u_{1j} = 1$ and $w_{2j} = u_{2j} = 0$
- $C(x) = 3$.

For example 2

- $w_{1j} = u_{1j} = 1$ and $w_{2j} = u_{2j} = 1$
- $C(x) = 4$

The monotonicity is respected.

If the priority relationship between the criteria is a linear ordering (one criteria in each category) then we can obtain a monotonic prioritized averaging (PA) operator

Prioritized Averaging Operators

Problem Formulation

- Collection of criteria partitioned into q distinct categories

  $H_1, H_2, ..., H_q$

- $H_i = \{C_i\}$: **One criteria** in criteria in category H_i.
- A prioritization between these categories

  $C_1 > C_2, ... > C_q$.

- Criteria C_i has higher priority than C_k if $i < k$.

Prioritized Averaging Operators
PA Operator

- Alternative $x \in X$
- $C_i(x) \in [0, 1]$ is x satisfaction to criteria C_i
- C(x) overall score for alternative x
- **P**rioritized **A**veraging (PA) operator

$$C(x) = \sum_{i=1}^{q} w_i C_i(x)$$

The w_i depend on $C_k(x)$ for $k < i$

Determination of Weights

- For category H_i we calculate $S_i = C_i(x)$
- S_i is the value of the least satisfied criteria in category H_i
- $S_0 = 1$ by convention
- Calculate

$$T_i = \prod_{k=1}^{i-1} S_k \qquad (T_3 = S_0 S_1 S_2)$$

$$u_i = T_i \qquad \text{(pre-weights)}$$

$$w_i = \frac{T_i}{T} \qquad T = \sum_i T_i$$

Prioritized Averaging Operator

$$C(x) = \sum_{i=1}^{q} w_i C_i(x)$$

$$w_i = \frac{T_i}{T} \qquad T = \sum_i T_i$$

$$T_1 = 1$$

$$T_i = C_1(x) C_2(x) C_3(x) \ldots C_{i-1}(x) \qquad i > 1$$

Weights decrease as i increases

Lack of satisfaction to higher priority criteria blocks compensation by lower priority criteria

Illustration

$$C_1 > C_2 > C_3 > C_4$$

$$C_1(x) = 1 \quad C_2(x) = 0.5 \quad C_3(x) = 0.2 \quad C_4(x) = 1$$

$T_1 = 1 \quad T_2 = 1 \quad T_3 = 0.5 \quad T_4 = 0.1$ **T = 2.6**

$$w_1 = 0.38 \quad w_2 = 0.38 \quad w_3 = 0.2 \quad w_4 = 0.04$$

$$C(x) = (0.38)(1) + (0.38)(0.5) + (0.2)(0.2) + (0.04)(1) = 0.65$$

$$C_1(y) = 0.2 \quad C_2(y) = 0.5 \quad C_3(y) = 1 \quad C_4(y) = 1$$

$T_1 = 1 \quad T_2 = 0.2 \quad T_3 = 0.1 \quad T_4 = 0.1$ **T = 1.4**

$$w_1 = 0.72 \quad w_2 = 0.14 \quad w_3 = 0.07 \quad w_4 = 0.07$$

$$C(y) = (0.72)(0.2) + (0.14)(0.5) + (0.07)(1) + (0.07)(1) = 0.35$$

Alternative Determination of S_i

$$H_i = \{C_{i1}, C_{i2}, C_{i3}, \ldots\ldots, C_{in_i}\}$$

S_i is effective satisfaction of criteria in H_i

$$S_i = \mathrm{Min}_j[C_{ij}(x)]$$ (Least satisfied criteria)

$$S_i = \sum_{j=1}^{n_i} \frac{1}{n_i} C_{ij}(x)$$ (Average satisfaction in H_i)

$$S_i = \mathrm{OWA}(C_{i1}(x), C_{i2}(x), C_{i2}(x), \ldots, C_{in_i}(x))$$

The End